Generalizations of Perelomov's identity on the completeness of coherent states

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We proof the Perelomov identity for arbitrary 2D lattices using Fourier transformation. We further generalize it to situations where the origin does not coincide with a lattice site, and where the form of the exponential factor is reminiscent of magnetic wave functions in uniaxial rather than symmetric gauge.

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Introduction.—Studies of two-dimensional spin liquids[1–18], which were originally motivated by the problem of high $T_{\rm c}$ superconductivity, are enjoying a renaissance of interest in present days[19–24]. One of the reasons is the accumulation of numerical evidence for spin liquid phases in various two dimensional spin models, including the Hubbard model on a honeycomb lattice[25] and the next-nearest neighbor Heisenberg antiferromagnet on the square lattice, usually referred to as the J_1–J_2 model[26]. Another reason is that they constitute intricate examples of topological phases[27–32], which currently receive significant interest in the context of topological insulators[33–35].

In fact, the concept of topological order[27–30] was discovered in a two dimensional spin liquid, the (Abelian) chiral spin liquid (CSL)[2,4–6,11,16,23,24,36]. The idea of this spin $s=\frac{1}{2}$ liquid, due to D.H. Lee[37], is to describe spin flip operators S_i^+ in a background of down spins by a bosonic quantum Hall wave function at Landau level filling factor $\nu=\frac{1}{2}$. Kalmeyer and Laughlin[6] discovered that this wave function, when supplemented by an appropriate gauge factor $G(z)=\pm 1$, is a spin singlet. As first pointed out by Zou, Douçot, and Shastry[5], the proof relies on an identity established by A.M. Perelomov in 1971 in the context of the completeness of systems of coherent states[38]. The Abelian CSL is the simplest example of a class of spin liquids, which are constructed using Landau level wave functions in ficticious or auxiliary magnetic fields. More recent examples of this class include the spin $S=1$ chirality liquid[9] (which is constructed via Schwinger boson projection form two Abelian CSLs with opposite chirality), the spin s non-Abelian CSL[15,24] (which supports spinon excitations with SU(2) level $k=2s$ statistics), and a hierarchy of spin liquid states[22] (which suggests that spinons in parity and time reversal invariant antiferromagnets with integer spin $s=2$ and higher obey SU(2) level $k=s$ non-Abelian statistics).

All the spin liquids in this class share two features. First, the mechanism of fractional quantization yielding spinon (and holon) excitations, is both mathematically and conceptually similar to the mechanism of fractional quantization in quantized Hall states. The fractional quantum number in the spin liquids is the spin $s=\frac{1}{2}$ of the spinon, which is fractional in the context of Hilbert spaces built out of spin flips, which carry spin one. The Abelian CSL is related to a Laughlin state in the quantum Hall system, while the family of non-Abelian CSLs are reminiscent of the Moore–Read[39,40] and Read–Rezayi states[41]. Second, many analytical results available for these highly complex states, including the singlet property for Abelian[5,6] and non-Abelian CSL states[15], as well as the recent construction of a parent Hamiltonian[24] for the non-Abelian CSL states, rely on Perelomov's identity[38].

This identity was originally derived from the properties of the Jacobi theta functions, and used to show that there is only one linear relation between certain systems of coherent states. In this Brief Report, we show that Perelomov's identity can be generalized or reformulated in several ways, which are highly expedient for applications to spin liquids.

To be precise, we do three things. First, we proof that the identity holds for arbitrary 2D lattices with one site per unit cell using Fourier transformation. Second, we generalize the identity to situations where the origin does not coincide with a lattice site. Third, we rewrite the identity such that the form of the exponential factor is reminiscent of magnetic wave functions in uniaxial rather than symmetric gauge. The last result is particularly useful when the spin liquids are formulated on lattices with periodic boundary conditions.

The Perelomov identity.—Consider a lattice spanned by $\eta_{n,m}=na+mb$ in the complex plane, with n and m integer and the area of the unit cell Ω spanned by the primitive lattice vectors a and b set to 2π,

$$\Omega=\left|\Im(a\bar{b})\right|=2\pi, \tag{1}$$

where $\Im$ denotes the imaginary part. Let $G(\eta_{n,m})=(-1)^{(n+1)(m+1)}$. Then

$$\sum_{n,m}P(\eta_{n,m})G(\eta_{n,m})e^{-\frac{1}{4}|\eta_{n,m}|^2}=0 \tag{2}$$

for any polynomial P of $\eta_{n,m}$.

Proof.—It is sufficient to proof the identity for the generating functional

$$\sum_{n,m} e^{\frac{1}{2}\eta_{n,m}\bar{z}} G(\eta_{n,m}) e^{-\frac{1}{4}|\eta_{n,m}|^2} = 0. \tag{3}$$

Since $G(\eta_{n,m})$ takes the value -1 on a lattice with twice the original lattice constants, we may rewrite this as

$$\sum_{n,m} e^{\frac{1}{2}\eta_{n,m}\bar{z}} e^{-\frac{1}{4}|\eta_{n,m}|^2} - 2\sum_{n,m} e^{\eta_{n,m}\bar{z}} e^{-|\eta_{n,m}|^2} = 0. \tag{4}$$

Kalmeyer and Laughlin[6] observed that for the square lattice, the second sum in (4) can be expressed as a sum of the Fourier transform of the function we sum over in the first term. We demonstrate here that their proof can be extended to arbitrary lattices.

To begin with, we define the Fourier transform in complex coordinates

$$\tilde{f}(\zeta) = \int d^2\eta f(\eta) e^{i\Re(\eta\bar{\zeta})}, \tag{5}$$

where $\Re$ denotes the real part and we have used (1). Since the area of the unit cell of our lattice is taken to be 2π, the reciprocal lattice is given by the original lattice rotated by $\frac{\pi}{2}$ in the plane without any rescaling of the lattice constants. In complex coordinates,

$$\zeta_{n',m'} = i(n'a + m'b), \tag{6}$$

as this immediately implies

$$\begin{aligned} \boldsymbol{R}_{n,m}\cdot\boldsymbol{K}_{n',m'} &= \Re(\eta_{n,m}\bar{\zeta}_{n',m'}) = \\ &= \Re\left((na+mb)(-i)(n'\bar{a}+m'\bar{b})\right) \\ &= nm'\Im(a\bar{b}) + mn'\Im(b\bar{a}) \\ &= 2\pi\cdot\text{integer}. \end{aligned}$$

Then

$$\sum_{n',m'} \tilde{f}(\zeta_{n',m'}) = \Omega \sum_{n,m} f(\eta_{n,m}). \tag{7}$$

Eq. (7) follows directly from

$$\sum_{n',m'} e^{i\Re(\eta\bar{\zeta}_{n',m'})} = \Omega\sum_{n,m}\delta^{(2)}(\eta_{n,m}-\eta), \tag{8}$$

which is just the 2D equivalent of the (Dirac comb) identity

$$\sum_{n'=-\infty}^{\infty} e^{2\pi i n' x} = \sum_{n=-\infty}^{\infty}\delta(x-n) \tag{9}$$

The r.h.s. of (9) is obviously zero if x is not an integer, and manifestly periodic in x with period 1. To verify the normalization, observe that since for any N odd,

$$\sum_{n'=-\frac{N-1}{2}}^{+\frac{N-1}{2}} e^{2\pi i n' y/N} = \begin{cases} N & \text{for } y = N\cdot\text{integer} \\ 0 & \text{otherwise.}\end{cases}$$

This implies

$$\frac{1}{N}\sum_{y=-\frac{N-1}{2}}^{+\frac{N-1}{2}}\ \sum_{n'=-\frac{N-1}{2}}^{+\frac{N-1}{2}} e^{2\pi i n' y/N} = 1,$$

which in the limit $N\to\infty$ is equivalent to

$$\int_{-\frac{N}{2}}^{+\frac{N}{2}} \frac{dy}{N} \sum_{n'=-\frac{N-1}{2}}^{+\frac{N-1}{2}} e^{2\pi i n' y/N} = 1$$

Substituting $x = y/N$ yields

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} dx \sum_{n'=-\infty}^{\infty} e^{2\pi i n' x} = 1,$$

which proves the normalization in (9).

We proceed by evaluation of the Fourier transform of $f(\eta) = e^{\frac{1}{2}\eta\bar{z}}e^{-\frac{1}{4}|\eta|^2}$:

$$\begin{aligned} \tilde{f}(\zeta) &= \int d^2\eta\, e^{\frac{1}{2}\eta\bar{z}} e^{-\frac{1}{4}|\eta|^2} e^{i\Re(\eta\bar{\zeta})} \\ &= \int d^2\eta\, e^{\frac{1}{2}\eta\bar{z}} e^{-\frac{1}{4}|\eta|^2} e^{\frac{i}{2}(\eta\bar{\zeta}+\bar{\eta}\zeta)} \\ &= 4\pi e^{-|\zeta|^2 + i\zeta\bar{z}} \end{aligned} \tag{10}$$

where we have used the integral

$$\begin{aligned} &\int d^2\eta\ F(\eta)\ e^{-\frac{1}{\alpha}(|\eta|^2-\bar{\eta}w)} \\ &\quad= F(\alpha\partial_{\bar{w}})\int d^2\eta\, e^{-\frac{1}{\alpha}(|\eta|^2-\bar{\eta}w-\eta\bar{w})}\Big|_{\bar{w}=0} \\ &\quad= F(\alpha\partial_{\bar{w}})\int d^2\eta\, e^{-\frac{1}{\alpha}(|\eta-w|^2-w\bar{w})}\Big|_{\bar{w}=0} \\ &\quad= \alpha\pi\, F(\alpha\partial_{\bar{w}}) e^{\frac{1}{\alpha}w\bar{w}} \ =\ \alpha\pi\, F(w)\Big|_{\bar{w}=0} \end{aligned}$$

with $F(\eta) = e^{\frac{1}{2}\eta\bar{z}+\frac{i}{2}\eta\bar{\zeta}}$, $\alpha = 4$, and $w = 2i\zeta$.

Substituting (10) into (7) we obtain

$$\sum_{n,m} f(\eta_{n,m}) = 2\sum_{n',m'} e^{-|\zeta_{n',m'}|^2 + i\zeta_{n',m'}\bar{z}} \tag{11}$$

If we now substitute $n' = -n$, $m' = -m$, and hence $i\zeta_{n',m'} = \eta_{n,m}$ into the r.h.s. of (11), we obtain (4). This completes the proof.

Generalization to lattices where the origin does not coincide with a lattice site.—We now assume a shifted lattice with the sites given by

$$\eta_{n,m} = na + mb + c, \tag{12}$$

where n and m are integer and a, b, and c are complex numbers such that the area of the unit cell Ω spanned

by the primitive lattice vectors a and b remains set to 2π (see (1) above). Then for any polynomial P of $\eta_{n,m}$,

$$\sum_{n,m} P(\eta_{n,m}) G(\eta_{n,m}) e^{-\frac{1}{4}|\eta_{n,m}|^2} = 0, \qquad (13)$$

where the gauge factor is now given by

$$G(\eta_{n,m}) = (-1)^{(n+1)(m+1)} e^{-\frac{1}{2}\Im(\eta_{n,m}\bar{c})}. \qquad (14)$$

Proof.—With $\eta'_{n,m} = \eta_{n,m} - c$, we write the exponential in (13) as

$$\begin{aligned} e^{-\frac{1}{4}|\eta_{n,m}|^2} &= e^{-\frac{1}{4}|\eta'_{n,m}|^2} e^{-\frac{1}{4}(\eta'_{n,m}\bar{c}+\bar{\eta}'_{n,m}c)}\, e^{-\frac{1}{4}|c|^2} \\ &= e^{-\frac{1}{4}|\eta'_{n,m}|^2} e^{\frac{1}{4}(\eta'_{n,m}\bar{c}-\bar{\eta}'_{n,m}c)}\, e^{-\frac{1}{2}\eta'_{n,m}\bar{c}}\, e^{-\frac{1}{4}|c|^2} \\ &= e^{-\frac{1}{4}|\eta'_{n,m}|^2} e^{\frac{1}{2}\Im(\eta_{n,m}\bar{c})}\, e^{-\frac{1}{2}\eta'_{n,m}\bar{c}}\, e^{-\frac{1}{4}|c|^2} \end{aligned}$$

If we now absorb the last two factors into the Polynomial $P(\eta_{n,m})$, (13) with (14) reduces to (2).

Comment.—For most applications, it is convenient to write (14) as

$$G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}\, e^{\frac{1}{2}[(a'n+b'm)c''-(a''n+b''m)c']}, \qquad (15)$$

where $a = a' + \imath a''$, $b = b' + \imath b''$, $c = c' + \imath c''$ and a', a'', etc. are real.

The Perelomov identity in uniaxial gauge.—The generalized identity (13) with (14) can further be rewritten as

$$\sum_{n,m} P(\eta_{n,m}) G(\eta_{n,m}) e^{-\frac{1}{2}\Im(\eta_{n,m})^2} = 0, \qquad (16)$$

with the gauge factor now given by

$$G(\eta_{n,m}) = (-1)^{(n+1)(m+1)} e^{\frac{1}{2}\Re(\eta_{n,m}-c)\Im(\eta_{n,m}+c)}. \qquad (17)$$

Proof.—If we substitute

$$P(\eta_{n,m}) \to P(\eta_{n,m})\, e^{+\frac{1}{4}\eta_{n,m}^2}$$

into (13), we obtain for the product of all the exponential factors

$$\begin{aligned} &e^{-\frac{1}{2}\Im(\eta_{n,m}\bar{c})}\, e^{+\frac{1}{4}(\eta_{n,m}^2-|\eta_{n,m}|^2)} \\ &\quad= e^{-\frac{1}{2}\Im(\eta_{n,m}\bar{c})}\, e^{+\frac{1}{4}\eta_{n,m}(\eta_{n,m}-\bar{\eta}_{n,m})} \\ &\quad= e^{-\frac{1}{2}\Im(\eta_{n,m}\bar{c})}\, e^{+\frac{1}{2}[\Re(\eta_{n,m})+\imath\Im(\eta_{n,m})]\Im(\eta_{n,m})} \\ &\quad= e^{-\frac{1}{2}\Im(\eta_{n,m}\bar{c})}\, e^{+\frac{1}{4}\Im(\eta_{n,m}^2)}\, e^{-\frac{1}{2}\Im(\eta_{n,m})^2} \\ &\quad= e^{+\frac{1}{4}\Im[\eta_{n,m}(\eta_{n,m}-2\bar{c})]}\, e^{-\frac{1}{2}\Im(\eta_{n,m})^2} \\ &\quad= e^{-\frac{1}{4}\Im(\bar{c}^2)}\, e^{+\frac{1}{4}\Im[(\eta_{n,m}-\bar{c})^2]}\, e^{-\frac{1}{2}\Im(\eta_{n,m})^2} \\ &\quad= e^{+\frac{1}{4}\Im(c^2)}\, e^{\frac{1}{2}\Re(\eta_{n,m}-c)\Im(\eta_{n,m}+c)}\, e^{-\frac{1}{2}\Im(\eta_{n,m})^2}. \end{aligned}$$

If we absorb the first factor into the polynomial, we obtain (16) with (17).

Comment.—For most applications, it is convenient to write (17) as

$$G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}\, e^{\frac{1}{2}(a'n+b'm)(a''n+b''m+2c'')}. \qquad (18)$$

For a rectangular lattice with $a'' = b' = c'' = 0$, this gauge factor reduces to

$$G(\eta_{n,m}) = (-1)^{n+m+1}. \qquad (19)$$

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